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THE CONSISTENCY STRENGTH OF "EVERY STATIONARY SET REFLECTS"

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ABSTRACT
The consistency strength of a regular cardinal so that every stationary set reflects is the same as that of a regular cardinal with a normal ideal $\mathscr{I}$ so that every $\mathscr{I}$-positive set reflects in a $\mathscr{I}$-positive set. We call such a cardinal a *reflection cardinal* and such an ideal a *reflection ideal*. The consistency strength is also the same as the existence of a regular cardinal κ so that every κ-free (abelian) group is κ^+-free. In L, the first reflection cardinal is greater than the first greatly Mahlo cardinal and less than the first weakly compact cardinal (if any).

The question we consider in this paper was raised by trying to find the equiconsistency strength of the existence of a regular cardinal κ so that κ-free abelian group is κ^+-free. Let κ be the least such cardinal. By a well known argument, κ is either the successor of a singular cardinal or an inaccessible cardinal. If κ is the successor of a singular cardinal then there are inner models which have measurable cardinals. Since weakly compact cardinals have the desired property, the search for the least consistency strength can focus on a regular limit cardinals. Although the question was asked for abelian groups, the problem and the solution are the same if instead we consider groups or transversals of families of countable sets.

The key notion in our considerations is the notion of reflection for a stationary set. A stationary subset $S \subseteq \kappa$ *reflects* if there is some limit ordinal $\gamma < \kappa$ so that $S \cap \gamma$ is stationary in γ. If there is a stationary subset of κ which does not reflect then by standard constructions (e.g. [Ek]), we can build a κ-free

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abelian group which is not κ^+-free. So the consistency strength will be at least that of a cardinal such that every stationary set reflects. As well, it is easy to see (and well known) that if every stationary set reflects in a regular cardinal then every κ-free abelian group is κ^+-free. So the problem reduces to two set-theoretic problems, the consistency strength of a regular cardinal in which every stationary set reflects and the consistency strength of a regular cardinal in which every stationary set reflects in a regular cardinal. Fortunately these two principles have the same consistency strength. In fact, $0^\#$ does not exist implies that if κ is a regular cardinal such that every stationary subset of κ reflects, then every stationary subset of κ reflects in a regular cardinal.

There has been quite a bit of work related to this problem. Kunen [Ku] showed that it is consistent that there is a regular cardinal which is not weakly compact such that every stationary set reflects. However, in Kunen's model the cardinal is weakly compact in L. So his proof requires the consistency of a weakly compact cardinal. Baumgartner [Ba] showed that it is consistent, assuming the consistency of a weakly compact cardinal, that every stationary subset of $\aleph_2$ consisting of ordinals of cofinality ω reflects. Later in [HaSh], it was shown that this statement is equiconsistent with the existence of a Mahlo cardinal. So there was hope that the consistency strength of the statement of a regular cardinal such that every stationary set reflects is less than that of a weakly compact cardinal. It should be noted that there is a limit to the results that can be proved. Magidor [Ma] has shown that the existence of a regular cardinal such that every two stationary sets reflect in a common ordinal is equiconsistent with the existence of a weakly compact cardinal. (In [Ma] the result is stated for stationary subsets of $\aleph_2$ consisting of ordinals of cofinality ω, but the proof establishes the result we have stated.)

One might also want to get many cardinals in which every stationary set reflects (without assuming the consistency of weakly compact cardinals). The idea would be to iterate the forcing that we use here. Not surprisingly the forcing can be iterated a finite number of times. However the existence of $\aleph_0$ regular cardinals such that every stationary set reflects implies the consistency of weakly compact cardinals. More precisely Magidor (unpublished) has proved the following result.

> If $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ are regular cardinals so that any stationary set consisting of ordinals of cofinality ω_1 (or any larger cofinality) reflects then all but finitely many of $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ are weakly compact in L.

A related question which we will not deal with here is full reflection. A set S

fully reflects in T if for every stationary $S' \subseteq S$, $\{\delta \in T : S' \cap \delta$ is not stationary$\}$ is non-stationary. Magidor [Ma] showed that it is equiconsistent with the existence of weakly compact cardinal that $\{\alpha < \omega_2 : \mathrm{cf}\,\alpha = \omega\}$ full reflects in $\{\alpha < \omega_2 : \mathrm{cf}\,\alpha = \omega_1\}$. For more on equiconsistency results concerning full reflection see [JeSh].

Suppose for the moment that we have a cardinal κ in which every stationary set reflects. Then the non-stationary ideal witnesses that κ has the following property.

> There is a proper κ-complete normal ideal $\mathscr{I}$ on κ such that if $X \in \mathscr{I}^+$ then $\{\alpha : X \cap \alpha$ is stationary in $\alpha\} \in \mathscr{I}^+$

We call such a cardinal a *reflection cardinal* and an ideal as above a *reflection ideal*. Here $\mathscr{I}^+$ denotes the $\mathscr{I}$-positive sets (i.e. the sets not in $\mathscr{I}$). A reflection ideal has the property that every positive set reflects in a positive set.

If a cardinal is a reflection cardinal, it is possible to give a more concrete description of a reflection ideal. Define a sequence of normal ideals on κ. Let $\mathscr{I}_0$ be the normal ideal generated by the sets which do not reflect in any limit ordinal. (Notice that $\mathscr{I}_0$ contains the non-stationary ideal.) If $\mathscr{I}_\alpha$ has been defined then let $\mathscr{I}_{\alpha+1}$ be the least normal ideal extending $\mathscr{I}_\alpha$ which contains every set X such that $\{\beta : X \cap \beta$ is stationary in $\beta\}$ is in $\mathscr{I}_\alpha$. At limit ordinals take the normal ideal generated by the union. Let $\mathscr{I} = \bigcup_{\alpha<\kappa^+} \mathscr{I}_\alpha$. Then κ is a reflection cardinal if and only if $\mathscr{I}$ is a proper ideal. In that case $\mathscr{I}$ is a reflection ideal.

Proposition 1. *There is a Π^1_1 formula which defines being a reflection cardinal.*

Proof. The statement that κ is not in the set $\mathscr{I}$ defined above is Π^1_1. □

Corollary 2. *If μ is weakly compact then μ is the limit of reflection cardinals.* □

Theorem 3. *If κ is a reflection cardinal, then κ is a reflection cardinal in* L.

Proof. Let $\mathscr{I}$ be the ideal described above and $\mathscr{J}$ be the ideal defined in the same way in L. Since $\mathscr{J} \subseteq \mathscr{I}$, $\mathscr{J}$ is a proper ideal. So κ is a reflection cardinal in L. □

There is another large cardinal notion which is similar to the notion of a reflection cardinal, namely that of a *greatly Mahlo cardinal*. A cardinal is greatly Mahlo if there is a proper κ-complete normal ideal $\mathscr{I}$ on κ such that $\mathscr{I}$

concentrates on regular cardinals and if $X \in \mathscr{I}^*$ then $\{\alpha : X \cap \alpha$ is stationary in $\alpha\} \in \mathscr{I}^*$. Here $\mathscr{I}^*$ denotes the dual filter. We can also describe greatly Mahlo as one of a family of notions of Mahlo cardinals. Fix κ an inaccessible cardinal. Define a sequence of subsets of κ as follows. A_0 is the set of regular cardinals less than κ. If A_α has been defined let $A_{\alpha+1} = \{\beta \in A_\alpha : A_\alpha \cap \beta$ is stationary in $\beta\}$. At limit ordinals δ of cofinality $\neq \kappa$, let $A_\delta = \bigcap_{\alpha<\delta} A_\alpha$. Finally if δ is a limit ordinal of cofinality κ, choose $(\alpha_\nu : \nu < \kappa)$ an increasing sequence cofinal in δ. Let A_δ be the diagonal intersection of $(A_{\alpha_\nu} : \nu < \kappa)$. Note that the sets A_α are uniquely determined modulo the non-stationary ideal and modulo the non-stationary ideal the sequence $(A_\alpha : \alpha < \kappa^+)$ is decreasing. The cardinal κ is said to be α-Mahlo if A_β is stationary for all $\beta < \alpha$. κ is greatly Mahlo if and only if each of the sets A_α is stationary. We will have occasion to refer to these sets relative to other cardinals μ. In which case we will denote by $A_{\beta,\mu}$ the set A_β defined by the same rules relative to μ.

THEOREM 4. *Assume* (V = L). *Suppose that* κ *is at most the first greatly Mahlo cardinal. Then* κ *is not a reflection cardinal.*

PROOF. For $\mu < \kappa$ define sequences of ordinals α_i, β_i as follows. Let α_0 be minimal and β_0 be minimal for α_0 so that

$$L_{\alpha_0} \models \text{“}\mu \text{ is not } \beta_0\text{-Mahlo”}.$$

Note that since μ is not greatly Mahlo, α_0 and β_0 are defined and less than μ^+. In general if α_i has been defined let α_{i+1} be the least ordinal if any and β_{i+1} minimal for α_{i+1} so that

$$L_{\alpha_{i+1}} \models \text{“}\mu \text{ is not } \beta_{i+1}\text{-Mahlo and } \beta_{i+1} < \beta_i.\text{”}$$

Since the sequence $\beta_0, \ldots, \beta_i$ is decreasing there is some n_μ where the sequence terminates. Let $n = n_\mu$. We claim that for each μ there is a closed unbounded set C such that if a regular cardinal $\lambda \in C$ then $n_\lambda > n$. Choose a club $(M_\gamma : \gamma < \mu)$ of elementary submodels of L_{α_n}. Let $C = \{\mu^{M_\gamma} : \gamma < \mu\}$. If $\lambda \in C$, then $n_\lambda \geqq n$. We now must see that for all but a non-stationary set of λ, λ is not $(\pi(\beta_n) - 1)$-Mahlo where π is the Mostowski collapse of M. (Note that β and, hence, $\pi(\beta)$ are always successor ordinals.)

This fact is an immediate consequence of the following claim. Before we can state the claim we need to point out an easy fact about L. Suppose that μ is a cardinal and $\beta < \mu^+$. If for $\mu, \beta < \alpha$

$$L_\alpha \models \text{“}\beta < \mu^+\text{”}$$

and $M \prec L_\alpha$ is such that $M \cap \mu = \gamma$ where γ is a cardinal and $\beta \in M$, then $\pi(\beta)$ is independent of α and M. Here π denotes the Mostowski collapse. This fact lets us speak about $\pi(\beta)$ without specifying M if μ and γ is understood. If there is a danger of ambiguity we will write π_μ or $\pi_{\mu\gamma}$.

CLAIM 4.1. *Suppose that μ is β-Mahlo.*

(a) *Then for all $\alpha < \beta$ and almost all $\gamma \in A_{\alpha\mu}$, γ is $\pi(\alpha)$-Mahlo.*

(b) *For almost all γ, is γ if $\pi(\beta)$-Mahlo then $\gamma \in A_{\beta\mu}$.*

When we assert that something holds for almost all ordinals less than some regular cardinal, we mean that there is a club in that cardinal such that the statement holds.

PROOF (of Claim). The proof is by induction on μ and for fixed μ by induction on β. The result is trivial for $\beta = 1$ and at limit ordinals. So we can assume that we have the result for ρ and $\beta = \rho + 1$. We first show (a). Let $(M_\rho : \rho < \mu)$ be an increasing continuous chain of elementary submodels of L_{μ^+} such that $\beta \in M_0$ and each M_γ has cardinality $< \mu$. There is a club D so that for each $\gamma \in D$, $M_\gamma \cap \mu = \gamma$. Further, for each $\alpha < \rho$ we can fix a club C_α as guaranteed by the inductive hypothesis and assume that if $\alpha \in M_\gamma$ then $C_\alpha \in M_\gamma$. (Of course, if we choose the least possible C_α then this statement is a consequence of the induction hypothesis.) From now on we will restrict our choice of ordinals to those in D.

Suppose now that γ is in $A_{\rho\mu}$ and for all $\alpha \in M_\gamma$ if $\alpha < \rho$ then $\gamma \in A_{\alpha\mu}$ and $A_{\alpha\mu}$ is stationary in γ. Consider $\delta < \pi(\rho)$. Choose $\alpha \in M_\gamma$ such that $\pi(\alpha) = \delta$. Since $\gamma \in A_{\alpha\mu}$, the induction hypothesis implies that γ is δ-Mahlo. By the induction hypothesis (b) there is a club C in γ so that if $\sigma \in C$ and σ is $\pi_\sigma(\delta)$-Mahlo then $\sigma \in A_{\delta\gamma}$. Suppose now that $\sigma \in C \cap C_\alpha \cap A_{\alpha\mu}$. Computing we have

$$\pi_{\mu\sigma}(\alpha) = \pi_{\gamma\sigma}(\pi_{\mu\gamma}(\alpha)) = \pi_{\gamma\sigma}(\delta).$$

So by (b) of the induction hypothesis, $\sigma \in A_{\sigma,\gamma}$. Hence γ is $\pi(\rho)$-Mahlo.

Next we consider part (b). Suppose γ is as above and γ is $\pi(\beta)$-Mahlo. Consider $\alpha < \beta$ such that $\alpha \in M_\gamma$. Let $\delta = \pi(\alpha)$. Let C be a club in γ so that if $\sigma \in C$ and $\sigma \in A_{\delta\gamma}$ then σ is $\pi_\gamma(\delta)$-Mahlo. Now consider $\sigma \in C \cap C_\alpha \cap A_{\delta\gamma}$. By the induction hypothesis, σ is $\pi_{\gamma\sigma}(\delta)$-Mahlo. Hence by computing the collapse maps it is $\pi_{\mu\sigma}(\alpha)$-Mahlo. Hence by the induction hypothesis, it is in $A_{\alpha\mu}$. So $A_{\alpha\mu}$ is stationary in γ, which completes the proof of the claim.

Given the claim we can partition the inaccessible cardinals $< \kappa$ into $\aleph_0$ pieces by letting $X_n = \{\mu : n_\mu = n\}$. It is easy to see that $X_n \in \mathscr{I}_{n+1}$. Finally by a theorem of Prikry and Solovay [PrSo], the singular ordinals in κ can be written as a diagonal sum of sets which only reflect in regular cardinals. So $\kappa \in \mathcal{I}_{\omega+1}$. □

COROLLARY 5. *The consistency strength of the existence of a reflection cardinal is strictly between that of a greatly Mahlo cardinal and a weakly compact cardinal.* □

We could prove by the same means that there is in L a hierarchy of greatly Mahlo cardinals below the first reflection cardinal. But we will not investigate this hierarchy here.

Now we want to prove that the notion we were looking for is that of a reflection cardinal. In the proof of the theorem we will use a lemma from [GiSh], which we state below.

LEMMA 6. *Suppose λ is a regular cardinal and $\mathbb{Q}$ is a notion of forcing which satisfies the λ-c.c. Suppose $\mathcal{I}$ is a normal λ-complete ideal on λ. For all $\mathcal{I}$-positive sets S and sequences of conditions $\bar{q} = (q_\alpha : \alpha \in S)$, there is a set C whose complement is in the ideal so that all $\alpha \in C \cap S$,*

$$q_\alpha \Vdash \text{“}\tau_{\bar{q}} \text{ is positive with respect to the ideal generated by } \mathcal{I}.\text{”}$$

Here $\tau_{\bar{q}}$ is the name for $\{\alpha : q_\alpha \in G\}$ where G is the $\mathbb{Q}$-generic set. □

This lemma is actually stated for the ideal of non-stationary sets, but the same proof works for any normal λ-complete ideal. Note as well that the ideal generated in the extension is just the closure of $\mathcal{I}$ under subsets and is a normal λ-complete ideal.

THEOREM 7. *If it is consistent that there is a reflection cardinal, then it is consistent that there is a cardinal so that every stationary set reflects at regular cardinal.*

PROOF. Since any reflection cardinal is also a reflection cardinal in L, we can assume that (V = L) and that κ is a reflection cardinal. Let $\mathcal{I}$ be a reflection ideal. Since we are working in L, $\mathcal{I}$-most ordinals are strongly inaccessible cardinals. We first do a preparatory forcing. The forcing will be an iteration with Easton supports of ($\tilde{Q}_\lambda : \lambda < \kappa$ and λ is inaccessible). We let $\tilde{Q}_\lambda$ be the name for the forcing which adds λ^+ Cohen subsets of λ. Denote this preparatory forcing as $\mathbb{R}$ and as usual let $\mathbb{R}_\lambda$ denote the poset resulting from the iteration up to λ, etc. Let G be an R-generic set.

We want to work in L[G] but first we must see that κ is still a reflection cardinal. Since the forcing is κ-c.c., the ideal consisting of the subsets of $\mathcal{I}$ is a normal ideal. For simplicity we will continue to refer to this ideal as $\mathcal{I}$ and hope that the context will make it clear which ideal we are referring to. It must be shown that this ideal is in fact a reflection ideal. Suppose that $\tilde{S}$ is the name for a set which is forced to be $\mathcal{I}$-positive. Let p be a fixed condition. For every possible α, choose p_α extending p so that $p_\alpha \Vdash \alpha \in \tilde{S}$. Let T be the set of such α. Since T is forced to contain $\tilde{S}$, $T \in \mathcal{I}^+$. Let W be the set of inaccessible cardinals such that for all $\lambda \in W$, $T \cap \lambda$ is stationary and for all $\alpha \in T \cap \lambda$, $p_\alpha \in \mathbb{R}_\lambda$. Since $\mathcal{I}$ is a reflection ideal, $W \in \mathcal{I}^+$. Consider any $\lambda \in W$. By Lemma 6 applied to $\mathbb{R}_\lambda$ and the ideal of non-stationary sets, there is $q_\lambda \in \{p_\alpha : \alpha < \lambda\}$ which forces "$\{\alpha < \lambda : p_\alpha \in \tilde{G}_\lambda\}$ is stationary". Notice that q_λ forces in $\mathbb{R}_\lambda$ that "$\tilde{S} \cap \lambda$" is stationary. Then since the remainder of the iteration of R is λ^+-complete and so preserves stationary sets, q_λ forces in $\mathbb{R}$ that "$\tilde{S} \cap \kappa$" is stationary.

Now apply Lemma 6 to $\mathcal{I}$. There is λ so that q_λ forces "$\{\rho \in W : q_\rho \in \tilde{G}\}$ is $\mathcal{I}$-positive". So q_λ forces that "$\tilde{S}$ reflects in an $\mathcal{I}$-positive set". Since q_λ extends p and p was arbitrary we have shown that $\mathcal{I}$ is a reflection ideal in $L[G]$.

Now work in $L[G]$. We will define an iteration of length κ^+ with $<\kappa$-support which will put a closed unbounded set through the complement of every non-reflecting set. If we can show that the forcing does not add any new functions from μ to κ for any $\mu < \kappa$, we will be left with an easy enumeration problem. We will define the iteration $\mathbf{P}_\alpha$ by induction. The posets will be contained in the set of functions, p, with $<\kappa$-support so that for all β, $p(\beta)$ is a closed bounded subset of κ. At stage α, we will be given a name $\tilde{S}_\alpha$ for a non-reflecting set and $\mathbf{P}_{\alpha+1}$ will be those functions p so that $p \restriction \alpha \in \mathbf{P}_\alpha$ and $p \restriction \alpha \Vdash$ "$p(\alpha)$ is disjoint from $\tilde{S}_\alpha$".

The key notion in the proof is that of a good ordinal for α. Consider α, let N be an expansion $(H(\kappa^{++}), \in)$ by the sequence used to define the forcing up to $\mathbb{P}_\alpha$. Fix a closed unbounded set $(M_\lambda : \lambda < \kappa)$ of elementary submodels of N so that if λ is inaccessible then ${}^{<\lambda}M_\lambda \subseteq M_\lambda$. We say that λ is *good* for α if for all $\beta \in M_\lambda \cap \alpha$ then there is a $\mathbb{P}_\beta \cap M_\lambda$-name for a closed unbounded set in $M_\lambda \cap \kappa$ disjoint from $\tilde{S}_\beta \cap M_\lambda$. Of course such a name will not be in M_λ. The definition of the good ordinals depends on the choice of the sequence of models but the set of good ordinals is uniquely determined modulo the non-stationary ideal. We say α is *nice* if $\mathcal{I}$-most ordinals are good for α. Notice that if λ is good for α, then, using the notation above, $\mathbb{P}_\alpha \cap M_\lambda$ is essentially λ-closed (i.e. contains a dense subset which is λ-closed). We give a justification of this statement below.

The proof is completed by proving two claims.

Claim 7.1. *For all $\alpha < \kappa^+$ and $\mu < \kappa$, if α is nice, then forcing with $\mathbb{P}_\alpha$ adds no functions from μ to κ.*

Proof (of Claim). Fix a sequence $(M_\lambda : \lambda < \kappa)$ as above such that $\tilde{f} \in M_0$ where $\tilde{f}$ is the name for a function from μ to λ. Let X be the set of inaccessible cardinals with the property that for all $\lambda \in X$, $M_\lambda \cap \lambda = \lambda$. (All but a non-stationary set of ordinals have this property.) By the hypothesis there is an inaccessible cardinal $\lambda > \mu$ which is good for α and such that $X \cap \lambda$ is stationary in λ. Choose $\bar{C} = (\tilde{C}_\beta : \beta \in M_\lambda \cap \alpha)$ a sequence of names for clubs as guaranteed by the fact that λ is good. Since there are two notions of forcing to consider, the large posets and their intersection with M_λ, we will let $\Vdash_\lambda$ denote the forcing relation for the restricted posets. Consider the structure $(M_\lambda, \bar{C}, \Vdash_\lambda)$. Choose an inaccessible cardinal $\rho < \lambda$ so that $(M_\rho, \bar{C} \cap M_\rho, \Vdash_\lambda) \prec (M_\lambda, \bar{C}, \Vdash_\lambda)$.

The point here is that looking at $M_\rho \cap \mathbb{P}_\alpha$ we have an iteration of essentially ρ-complete forcings. To see this note first that M_ρ is closed under sequences of length $< \rho$. As well if for some $\gamma < \rho$ we are given an increasing sequence $(p_\delta : \delta < \gamma)$ of conditions which satisfy for all $\beta \in \operatorname{dom} p_\delta$, $p_\delta \restriction \beta \Vdash_\lambda$ "$\tilde{C}_\beta$ is cofinal in $\sup p_\lambda(\beta)$", then the sequence has an upper bound. Again since M_ρ is closed under sequences of length $< \rho$ there is an M_ρ-generic set G for $\mathbb{P}_\alpha \cap M_\rho$. Further this generic set can be taken to have the additional property that if $\beta \in M_\rho \cap \alpha$ then

$$\{\gamma \colon \text{there is } p \in G \text{ such that } p \restriction \beta \Vdash \gamma \in \tilde{C}_\beta\}$$

is unbounded in ρ.

Having made this choice we can now take the union of the components of G. This is a sequence of closed unbounded sets in ρ. Let p denote these closed unbounded sets with ρ added. More formally p is the function whose domain is $\{\beta < \alpha : \beta \in M_\rho\}$ and $p(\beta) = \bigcup_{q \in G} q(\beta) \cup \{\rho\}$. We now prove by induction on β that $p \restriction \beta \in \mathbb{P}_\beta$ and $p \restriction \beta \Vdash \rho \notin \tilde{S}_\rho$. To see this note that the induction hypothesis implies that $p \restriction \beta \in \mathbb{P}_\beta \cap M_\lambda$. Since M_ρ is the restriction of an elementary submodel of $(M_\lambda, \bar{C}, \Vdash_\lambda)$, we have that $p \restriction \beta \Vdash_\lambda \rho \in \tilde{C}_\beta$, which completes the proof of the subclaim. Finally to complete the proof of the claim notice that p decides the value of the function (and forces it to be bounded by ρ).

Claim 7.2 *For all $\alpha < \kappa^+$, α is nice.*

Proof (of Claim). The proof here is similar to the proof of the last claim. Only the successor case needs to be worried about since the limit cases are handled either by taking intersection or diagonal intersection. Assume the result for α. First we can fix a sequence of models $(M_\lambda : \lambda < \kappa)$ of the appropriate structure. Next we suppose that $X \in \mathscr{I}^+$ is a set of inaccessibles which is not good with respect to the chosen sequence and for $\lambda \in X$, $M_\lambda \cap \kappa = \lambda$. Choose $Y \in \mathscr{I}^+$ so that X reflects in Y and Y consists of inaccessible cardinals. Now choose $\lambda \in Y$ which is good for α (with respect to the sequence). Expand M_λ by a sequence of names $(\tilde{C}_\beta : \beta < \alpha)$ for the closed unbounded sets and the forcing at λ. Find $\rho \in X$ so that M_ρ is the restriction of an elementary submodel of the expanded version of M_λ.

Since forcing with $\tilde{R}^{\rho+1}$ adds no new subsets of ρ, it is enough to work in the extension we get by forcing $\mathbb{R}_\rho * \tilde{Q}_\rho$. Work first in $L[G_\rho]$. Here the interpretation of $\tilde{Q}_\rho$ is the poset for adding ρ^+ Cohen subsets of ρ. By adding ρ of these subsets we can assume that M_ρ, $\tilde{S}_\alpha \cap M_\rho$ and $(\tilde{C}_\beta \cap M_\rho : \beta < \alpha)$ are in $L[G_\rho]$. In this universe $M_\rho \cap \mathbb{P}_\alpha$ is essentially ρ-closed. Since we have added Cohen subsets of ρ to $L[G_\rho]$, we have added generic subsets for $M_\rho \cap \mathbb{P}_\alpha$. In fact every condition extends (in $L[G]$) to a generic set. Consider any such generic set which is in $L[G]$. As in the previous claim this set can be extended to a condition in $\mathbb{P}_\alpha$. As well this generic set (and so the condition) determines the value of $\tilde{S}_\alpha \cap \rho$. Since $\tilde{S}_\alpha$ is forced not to reflect there is a closed unbounded subset of ρ which does not intersect the decided value of $\tilde{S}_\alpha$. Since the remainder of the forcing does not destroy stationary subsets of ρ, such a club must exist in the generic extension of $L[G_\rho]$ by the Cohen subset of ρ. We have shown that every condition in $M_\rho \cap \mathbb{P}_\alpha$ is contained in a generic set, such that when we extend $L[G_\rho]$ by that generic set, the interpretation of $\tilde{S}_\alpha \cap M_\rho$ is not stationary. So there must exist a name for a $\mathbb{P}_\alpha \cap M_\rho$-name for a club which is disjoint from $\tilde{S}_\alpha \cap M_\rho$. But this shows that ρ is good for $\alpha + 1$ (with respect to the model M_ρ). This contradicts the choice of X and Y. □

There remains the question of how large the first cardinal such that every stationary set reflects can be. In [Ma], it is shown assuming the consistency of infinitely many supercompacts that it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects. As well in [MaSh], it is shown, assuming the same hypothesis, that it is consistent that every $\aleph_{\omega^2+1}$-free (abelian) group is $\aleph_{\omega^2+2}$-free. Both these results are the optimal consistent with ZFC. On the other hand if we don't want to use large cardinals the restrictions on the first cardinal such that every stationary set reflects.

PROPOSITION 8. *Assume* $\neg \exists 0^\#$. *If* κ *is a regular cardinal such that every stationary set reflects, then* κ *is* ω*-Mahlo.*

PROOF. It is enough to show that there is a stationary set of regular cardinals. Since $0^{\#}$ doesn't exist, κ is an inaccessible cardinal and so contains a closed unbounded set of cardinals. As well, if μ is a cardinal and regular in L, then μ is regular. Assume that there is no stationary set of regular cardinals. By the previous remark, κ contains a closed unbounded set consisting of cardinals which are singular in L. By [PrSo], there is a stationary set which does not reflect. (More exactly reflects only in a set of regular cardinals in L, which by our assumption is a non-stationary set.) $\square$

The optimal value for the first cardinal such that every stationary set reflects, assuming $\neg \exists 0^{\#}$, can be achieved.

THEOREM 9. *If it is consistent that there is a reflection cardinal then it is consistent that every stationary subset of the first ω-Mahlo cardinal reflects.*

PROOF. First force as above to get a cardinal κ so that every stationary set reflects. Define an iterated forcing of length κ with Easton support as follows. A condition p is a partial function which is trivial except at inaccessible cardinals. If λ is an inaccessible cardinal and $\lambda \in \operatorname{dom} p$ then $p(\lambda)$ is a $\mathbb{P} \restriction \lambda$-name for an element which is either the empty set or a pair $(\tilde{n}, \tilde{C})$ where $\tilde{n}$ is a natural number and λ is $\tilde{n}$-Mahlo (in the ground model) and $\tilde{C}$ is a closed bounded subset of λ disjoint from the n-Mahlo cardinals below λ. If n is a natural number we will denote by $(n, \tilde{C})$ the name for the pair whose first element is forced to be n and second element is forced to be $\tilde{C}$. Suppose λ is an inaccessible cardinal and consider the conditions which extend $\{(\lambda, (n, \varnothing))\}$. This can be viewed as a three stage iteration the first being λ-c.c. The next stage shoots a club through the ordinals which are not n-Mahlo and so preserves any stationary set which contains no n-Mahlo cardinals (in the extension). The last stage of the iteration is λ^{+}-closed.

From the comments above it is not hard to prove that the forcing preserves cardinals and cofinalities. Next we want to show that for any inaccessible cardinal $\lambda < \kappa$ if $p(\lambda) = (m, \tilde{C})$ and $n \leqq m$, then $p \Vdash \lambda$ is n-Mahlo. We prove this by induction on n. For $n = 0$ the result is clear. Suppose $n = k + 1$. Let $q \in \mathbb{P} \restriction \lambda$ be any condition extending $p \restriction \lambda$. For every $\lambda > \mu > \sup \operatorname{dom} q$, such that μ is k-Mahlo, let $r_\mu = q \cup \{(\mu, (k, \varnothing))\}$. By Lemma 6 and the induction hypothesis, some r_μ forces (in $\mathbb{P} \restriction \lambda$) that there is a stationary set of k-Mahlo cardinals below λ. Hence λ is forced by $\mathbb{P} \restriction \lambda$ that λ is $k + 1$-Mahlo. But the following stages of the iteration preserves this property.

A similar argument shows that κ itself is forced to be the first ω-Mahlo cardinal. It remains to see that it is forced that every stationary subset of κ reflects. This is proved much as above. Suppose $\tilde{S}$ is the name of a stationary subset of κ. Suppose q is a condition. For each possible α choose $p_\alpha \Vdash$ "$\alpha \in \tilde{S}$" and p_α extends q. By extending p_α further, we can assume that there is a natural number m_α such that $p_\alpha \Vdash$ "α is not m_α-Mahlo". Let T be the set of these ordinals. Clearly T is stationary (since it is forced to contain $\tilde{S}$). Choose a club C so that if $\beta \in T$, $\alpha \in C$ and $\beta < \alpha$ then sup dom $p_\beta < \alpha$. Choose an inaccessible cardinal λ so that $T \cap \lambda$ is stationary in λ. Choose m and a stationary subset T_1 of $T \cap \lambda$ so that for each $\alpha \in T_1$, $m_\alpha = m$. By Lemma 6, there is p_α which forces in $\mathbb{P} \restriction \lambda$ that $\{\beta \in T_1 : p_\beta \in G\}$ is stationary. Finally $p_\alpha \cup \{(\lambda, (m+1, \varnothing))\}$ forces in $\mathbb{P}$ that $\tilde{S} \cap \lambda$ is stationary in λ. □

The fact that in the forcing above preserves the property that every stationary set reflects could also be proved by appealing to Proposition 1.12 of [GiSh]. But the verification that the hypothesis of the proposition holds would be as long as the direct proof.

There is another related question namely trying to ensure that all stationary sets disjoint to a given costationary set reflect. The consistency of this statement implies the consistency of a Mahlo cardinal. In fact:

Theorem 10. *The existence of a cardinal such that every stationary set consisting of singular ordinals reflects in a regular cardinal is equiconsistent with the existence of a Mahlo cardinal.*

This theorem can be proved by methods similar to that of the proof of Theorem 7. But this proof can also be done without the use of a preparatory forcing, if we use the idea underlying [Sh1]. Since this proof may have independent interest, we will prove the following stronger theorem.

Theorem 11. *Suppose that κ is a strongly inaccessible cardinal and $E \subseteq \kappa$ is a stationary set consisting of regular cardinals. Further suppose that for all $\lambda \in E$, $\diamondsuit(\lambda)$ holds and $2^\kappa = \kappa^+$. Then there is a forcing notion $\mathbb{P}$, so that if we force with $\mathbb{P}$, E remains stationary, no new bounded subsets of κ are added and every stationary set which is disjoint to E reflects in an element of E.*

Proof. The forcing is an iteration of length κ^+ where we shoot a club through every set disjoint from E which does not reflect in a member of E. As before, for $\alpha < \kappa^+$ we have $\mathbb{P}_\alpha$ and $\tilde{S}_\alpha$ a $\mathbb{P}_\alpha$-name for a set which does not reflect in E. To prove the theorem it is enough to establish the following claim.

Claim 11.1. *Suppose $\alpha < \kappa^+$. Furthermore suppose $M \prec (\mathrm{H}(\kappa^{++}), \in)$,*

$M \cap \kappa = \lambda \in E$, $|M| = \lambda$, ${}^{\lambda}M \subseteq M$ and the sequences used to define the $\mathbb{P}_\alpha$ are elements of M. Then there exists (in the ground model) a set, G, which is M-generic for $M \cap \mathbb{P}_\alpha$. Also G is the set of restrictions of some condition in $\mathbb{P}_\alpha$.

PROOF (of Claim). We will first give the proof under the assumption that $\alpha < \lambda$ and then describe the modifications needed to prove the general case. First notice that if G is any set which is M-generic then G is the set of restrictions of a condition. To see this note that G consists of a sequence of clubs in λ, G determines the value of $\tilde{S}_\beta \cap \lambda$ (for $\beta < \alpha, \beta \in M$), and the βth club is disjoint from the determined value of $\tilde{S}_\beta$. Since λ is forbidden from being in any of the $\tilde{S}_\beta$, we can complete each club in λ to a closed set in κ by adding λ and so get a condition.

Notice as well that if H is an M-generic set for some $\mathbb{P}_\beta$, then since H can be completed to a condition and determines the value of $\tilde{S}_\beta \cap \lambda$, the determined value is non-stationary (as $\tilde{S}_\beta$ is forced to be non-reflecting). So there is a club C_β in λ which is disjoint from the determined value of $\tilde{S}_\beta$. The naive strategy for building a generic set should now be clear. We inductively choose M-generic sets G_β for $\mathbb{P}_\beta \cap M$. Then at successor stages we can use C_β to choose extend the generic set (i.e. we define a sequence of closed sets and at every stage make sure to include an element of C_β). The difficulty with this approach is that there is no guarantee that at limit stages we will have a generic set.

There is another approach, which would work if the forcing were essentially λ-complete. Namely we enumerate the dense sets in M in order type λ. Then we choose a λ sequence of conditions (which may be functions with domain α) and make sure that we meet every element of the dense. By using $\diamondsuit(\lambda)$, we can combine the two approaches.

To begin we enumerate as $(D_\gamma : \gamma < \lambda)$ the dense subsets of $\mathbb{P}_\beta$ which are in M, for all $\beta \leqq \alpha$. Let $\beta(\gamma)$ denote the β so that D_γ is a dense set of $\mathbb{P}_\beta$. Next choose disjoint stationary sets X_γ of λ so that $\diamondsuit(X_\gamma)$ holds for each $\gamma < \lambda$. Using $\diamondsuit(X_\gamma)$, predict for each $\delta \in X_\gamma$ a $\beta(\gamma)$-sequence, q_δ, of closed subsets of δ. If possible choose $r_\delta \in D_\gamma$ which extends q_δ. (Note that such an r_δ will exist if and only if q_δ is the restriction of a condition.) By restricting to a club (in λ) we can assume that for all γ and $\delta \in X_\gamma$, δ is $\geqq$ the supremum of

$$\{\sup\{\sup r_\rho(\tau) : \tau \in \operatorname{dom} \rho\} : \rho < \delta\}.$$

Now we can describe the construction. We will define the generic set G by induction on $\beta \leqq \alpha$, where we will produce a set G_β which is M-generic for $\mathbb{P}_\beta$. Suppose we have done the construction for β. Let C_β be a club which is forced by G_β to be disjoint from $\tilde{S}_\beta$. The construction is now carried out in stages to define a sequence, $(c_\rho : \rho)$ of closed subsets of λ which will give us $G_{\beta+1}$. For δ, we abuse notation and write $G_\beta \cap \delta$ for the sequence of intersections of the closed sets with δ. Suppose c_ρ has been defined and the greatest element of c_ρ is δ. There are two possibilities to consider. First suppose (and this is the most interesting case) that $G_\beta \cap \delta$ concatenated with $c_\rho \cap \delta$ is $q_\delta \restriction \beta + 1$ and that r_δ exists. In this case we let ζ be the least element of C_β greater than sup $r_\delta(\beta)$ and define $c_{\rho+1} = r_\delta(\beta) \cup \{\zeta\}$. Notice that if we can carry out the construction then $r_\delta \restriction \beta \in G_\beta$. So $c_{\rho+1}$ is forced by G_β to be disjoint from $\tilde{S}_\beta$. If we do not have r_δ as above then we just take ζ the least element of C_β greater than δ and let $c_{\rho+1} = c_\rho \cup \{\zeta\}$.

At limit ordinals we take unions and add the supremum. Since this supremum is in C_β, again we have a closed set which is forced by G_β to be disjoint from $\tilde{S}_\beta$. Since we will need to refer to this set of ordinals, let $W_\beta = \{\sup c_\rho : \rho < \lambda\}$.

To complete the proof we will show by induction on $\beta \leqq \alpha$ that G_β is M-generic. We will only do the limit case as the successor case is similar. Notice that the induction hypothesis implies G_β can be extended to a condition. Consider now any dense subset of $\mathbb{P}_\beta$ which is in M. I.e. consider D_γ where $\beta(\gamma) = \beta$. Notice that $W = \bigcap_{\sigma<\beta} W_\sigma$ is a club. By $\diamondsuit(X_\gamma)$, there is $\delta \in W \cap X_\gamma$ so that $q_\delta = G_\beta \cap \delta$. Since G_β can be extended to a condition, so can q_δ. Hence r_δ is defined. In the construction, we guaranteed $r_\delta \in G_\beta$. So $G_\beta \cap D_\gamma$ is non-empty.

We now describe the modifications necessary if $\alpha > \lambda$. Since $|\alpha| = \kappa$, there is a one-to-one and onto map $g \in M$ from α onto κ. (Note that $g \cap M$ is a one-to-one and onto map from $M \cap \alpha$ to λ.) We can replace our poset $\mathbb{P}_\alpha$ with an equivalent one. At stage β, rather than trying to force a club which is disjoint from $\tilde{S}_\beta$, we try to force a club subset of $[g(\beta), \kappa)$ which is disjoint to the set defined to be $\{g(\beta) + \nu : \nu \in \tilde{S}_\beta\}$. The effect of doing this is to guarantee that for a club of $\delta < \lambda$ and for any condition $p \in \mathbb{P}_\alpha$, $p \cap \delta$ is contained in a set of cardinality δ. So we can use $\diamondsuit$ as before. The other change in the proof is in the definition of W. In this case we want that $\delta \in W_\sigma$ for all σ such that $g(\sigma) < \delta$. Since this is a diagonal intersection, the proof can be carried out as before. □

In contrast with the difficulties involved in iterating the forcing which makes every stationary set reflect and the results in [JeSh], we can iterate the forcing above as long as complement of the sets all of whose subsets we want to reflect is large enough. For example, we can prove that for K a class of strongly inaccessible Mahlo cardinals such that $\mu = \sup(K \cap \mu) > \mathrm{cf}\,\mu$ implies $2^\mu = \mu^+$, there is a forcing extension which does not collapse cardinals nor change cofinalities such that for every $\lambda \in K$ any stationary subset of $\{\delta < \lambda : \delta$ is not inaccessible implies $\mathrm{cf}\,\delta > \mu$ for every $\mu \in K \cap \lambda\}$ reflects. The key point here is that when we consider the forcing as an iterated forcing the iterate from any λ on is essentially μ-complete for all $\mu < \lambda$. Another possibility is that we have the class K and a function h so that for all $\lambda \in K$, $\{\delta < \lambda : \delta$ is $h(\delta)$ Mahlo$\}$ is stationary. Then there is a forcing extension so that for every $\lambda \in K$, every stationary subset of $\{\delta < \lambda : \delta$ is inaccessible and δ is not $h(\lambda)$-Mahlo or δ is singular and $\mathrm{cf}\,\delta > \mu$ for all $\mu \in K \cap \lambda\}$. It is possible to get forcing extensions with stronger reflection properties if reflection cardinals are used.

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